

# Computer assisted proof of the existence of homoclinic tangency for the Hénon map and for the forced-damped pendulum

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## Abstract

We present a topological method for the efficient computer assisted verification of the existence of the homoclinic tangency which unfolds generically in a one-parameter family of planar maps. The method has been applied to the Hénon map and the forced damped pendulum ODE.

**Keywords:** homoclinic tangency, covering relation, cone condition, transversal intersection, computer assisted proof

## 1 Introduction

The goal of our paper is to describe a method for the verification of the existence of the quadratic homoclinic tangency which unfolds generically in a one-parameter family of planar maps. This is an important problem in dynamics, because establishing the existence of the generic homoclinic tangency has very deep dynamical consequences, see for example [PT, WY1, WY2, RC] and references given there.

Our paper was inspired by the works of Arai and Mischaikow [A, AM], who combined some tools from the Conley index theory, the tools of computational homology from *the CHomP project* [CHOMP] and the set oriented numerical methods from *the GAIO project* [GAIO], into a method for computer assisted proof of the existence of the generic homoclinic tangency. Using it they proved in

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[AM] the existence of the generic homoclinic tangency for the dissipative Hénon map  $H_{a,b}(x, y) = (a - x^2 + by, x)$  for parameter values close to  $a = 1.4, b = 0.3$  and  $a = 1.3, b = -0.3$ . Their method contains essentially two separate parts: first using the Conley index approach they prove the existence of the homoclinic tangency for some parameter value (steps 1 to 5 in the terminology used in [AM]) and in the second part (step 6 in [AM]) they verify some transversality-type condition, which implies the genericity of the homoclinic tangency established in the first part. The computation times reported in [AM] are around 260 minutes and 100 minutes on a PowerMac G5 (2GHz) for  $b = 0.3$  and  $b = -0.3$ , respectively. In these computations the second part took 61 minutes and 24 minutes, respectively. From these computation times it is quite clear that there is a little hope to apply successfully this method to ODEs.

Our method, also topological and geometric in spirit, is based on the observation that the computations done by Arai and Mischaikow in [AM] in the second part of their approach should be in principle sufficient to obtain the whole result, both the existence of the homoclinic tangency and its genericity. Obviously for this end, one should use different tools, ours are essentially those of differential topology, but developed earlier in the context of topological dynamics, like the covering relations combined with the cone conditions, see [KWZ, Z]. As a test case we give a computer assisted proof of the existence of the generic homoclinic tangency for the dissipative Hénon map with  $b = -0.3$  and  $a \approx 1.3145$  in the computation time 0.2 sec on the Intel Xeon 5160, 3GHz processor. This should be contrasted with the fact that in [AM] this case took around 100 minutes.

An application of our method to an ODE requires an efficient rigorous  $C^2$ -solver for ODEs. By this we mean an algorithm for rigorous integration of ODEs together with their variational equations up to the second order. Such an algorithm has been recently developed by the authors in [WZ2] and is now a part of the CAPD library [CAPD]. Using this algorithm we were able to prove the existence of the generic homoclinic tangency for the  $2\pi$ -shift along the trajectory of the periodically forced pendulum equation

$$\ddot{x} + \beta \dot{x} + \sin(x) = \cos(t). \quad (1)$$

for  $\beta \approx 0.2471$  (see Theorem 19 in Section 7). The computation time for this proof is 30 sec on the Intel Xeon 5160, 3GHz processor.

It should be mentioned that similar results for the Hénon map have been obtained by a complex analytic method of Fornaess and Gavosto [FG1, FG2]. Compared to their method, which depends on the analyticity of maps, our method and that of Arai and Mischaikow are rather geometric and topological, and are designed so that they can be applied to a wider class of maps. Essentially, we require a continuous family of  $C^2$  diffeomorphisms for which we can compute the image of the maps using interval arithmetic.

The content of the paper may be briefly described as follows: in Sections 2 and 3 following mainly [AM] we give basic definitions and restate the problem of the existence of the quadratic homoclinic tangency unfolding generically as the transversality question for a dynamical system induced by the given one on the

projective bundle. In Section 4 we discuss how this transversality problem can be solved using the covering relations linked with the cone conditions. Moreover, we derive computable estimates for the dependence on a parameter of (un)stable manifolds of the hyperbolic fixed point, which will be later used in the computer assisted proofs. In Section 5 we illustrate our approach on a toy example, with the intention that the reader may see and appreciate some details of the method, which are later hardly visible when we report on the computer assisted proof for the Hénon map and for the forced damped pendulum in Sections 6 and 7, respectively.

## 2 Homoclinic tangency and the projectivization

### 2.1 Invariant manifold - basic notation

**Definition 1** Consider the map  $f : X \supset \text{dom}(f) \rightarrow X$ .

Let  $x \in X$ . Any sequence  $\{x_k\}_{k \in I}$ , where  $I \subset \mathbb{Z}$  is a set containing 0 and for any  $l_1 < l_2 < l_3$  in  $\mathbb{Z}$  if  $l_1, l_3 \in I$ , then  $l_2 \in I$ , such that

$$x_0 = x, \quad f(x_i) = x_{i+1}, \quad \text{for } i, i+1 \in I$$

will be called an orbit through  $x$ . If  $I = \mathbb{Z}_-$ , then we will say that  $\{x_k\}_{k \in I}$  is a full backward orbit through  $x$ .

**Definition 2** Let  $X$  be a topological space and let the map  $f : X \supset \text{dom}(f) \rightarrow X$  be continuous.

Let  $Z \subset \mathbb{R}^n$ ,  $x_0 \in Z$ ,  $Z \subset \text{dom}(f)$ . We define

$$\begin{aligned} W_Z^s(z_0, f) &= \{z \mid \forall_{n \geq 0} f^n(z) \in Z, \quad \lim_{n \rightarrow \infty} f^n(z) = z_0\}, \\ W_Z^u(z_0, f) &= \{z \mid \exists \{x_n\} \subset Z \text{ a full backward orbit through } z, \text{ such that} \\ &\quad \lim_{n \rightarrow -\infty} x_n = z_0\}, \\ W^s(z_0, f) &= \{z \mid \lim_{n \rightarrow \infty} f^n(z) = z_0\}, \\ W^u(z_0, f) &= \{z \mid \exists \{x_n\} \text{ a full backward orbit through } z, \text{ such that} \\ &\quad \lim_{n \rightarrow -\infty} x_n = z_0\}. \end{aligned}$$

If  $f$  is known from the context, then we will usually drop it and use  $W^s(z_0)$ ,  $W_Z^s(z_0)$  etc., instead.

### 2.2 Projectivization of the dynamics

The notation and setting is the one used in [AM]. Let  $f : X \rightarrow X$  be a diffeomorphism of a manifold  $X$ .

**Definition 3** Let  $z_0 \in X$ . We say that  $z_0$  is a hyperbolic fixed point for  $f$  iff  $f(z_0) = z_0$  and  $Sp(Df(z_0)) \cap S^1 = \emptyset$ , where  $Df(z_0)$  is the derivative of  $f$  at  $z_0$ ,  $Sp(A)$  denotes the spectrum of a square matrix  $A$ .

We denote the tangent bundle of  $X$  by  $TX$  and the differential of  $f$  by  $Df$ . From the dynamical system  $f : X \rightarrow X$  we can derive a new dynamical system  $Pf : PX \rightarrow PX$  which is defined as follows. The space  $PX$  is the projective bundle associated to the tangent bundle of  $X$ , that is, the fiber bundle on  $X$  whose fiber over  $x \in X$  is the projective space of  $T_x X$ . That is,

$$PX = \bigcup_{x \in X} P_x X := \bigcup_{x \in X} \{\text{one-dimensional subspace of } T_x X\}.$$

For a submanifold  $S$  in  $X$  by  $PS$  we will denote its projectivization, which is given by

$$PS = \{(x, [v]) \in PX \mid x \in S, v \in T_x S \setminus \{0\}\}$$

It is easy to see that  $P(S)$  is a manifold and  $\dim(P(S)) = 2\dim(S) - 1$ .

Define  $Pf$  to be the map induced from  $Df$  on  $PX$ , namely,  $Pf(x, [v]) := (f(x), [Df(x) \cdot v])$  where  $0 \neq v \in T_x X$ ,  $[v]$  is the subspace in  $T_x X$  spanned by  $v$  and  $Df(x) : T_x X \rightarrow T_{f(x)} X$  is the derivative of  $f$  at  $x$ .

Let us identify  $X$  with the zero section of  $TX$ . We define the map  $\pi : TX \setminus X \rightarrow PX$  by  $\pi((x, v)) = (x, [v])$  for  $x \in X, v \in T_x X$ .

Let  $p \in X$  be a hyperbolic fixed point of  $f$  and let  $T_p X = \tilde{E}_p^s \oplus \tilde{E}_p^u$  be the corresponding splitting of the tangent space into stable and unstable subspaces for  $Df(p)$ . Define  $E_p^s := \pi(\tilde{E}_p^s \setminus \{0\})$  and  $E_p^u := \pi(\tilde{E}_p^u \setminus \{0\})$ .

### 2.3 Dimension two

**Theorem 1** *Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a diffeomorphism and let  $p$  be a hyperbolic fixed point with one dimensional stable and unstable manifolds.*

*Then*

1.  $(p, E_p^u)$  is a hyperbolic fixed point for  $Pf$  such that

- $W^u((p, E_p^u), Pf) = P(W^u(p, f)), \dim W^u((p, E_p^u), Pf) = 1,$
- $W^s((p, E_p^u), Pf) = \{(z, [v]) \in PX \mid z \in W^s(p, f), (z, [v]) \notin P(W^s(p, f))\},$   
 $\dim W^s((p, E_p^u), Pf) = 2.$

2.  $(p, E_p^s)$  is a hyperbolic fixed point for  $Pf$  such that

- $W^s((p, E_p^s), Pf) = P(W^s(p, f)), \dim W^s((p, E_p^s), Pf) = 1,$
- $W^u((p, E_p^s), Pf) = \{(z, [v]) \in PX \mid z \in W^u(p, f), (z, [v]) \notin P(W^u(p, f))\},$   
 $\dim W^u((p, E_p^s), Pf) = 2.$

**Proof:** It is easy to see that the stable and unstable sets are as stated. There remains for us to show the hyperbolicity, only. Consider first  $(p, E_p^u)$ . Let us change the coordinate system in  $\mathbb{R}^2$  such that  $p = 0$ ,  $Df(p)$  is diagonal

$$Df(p) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

where  $|\lambda| > 1$  and  $|\mu| < 1$ . From now on all considerations will be done in this coordinate frame.

Around  $(p, E^u(p))$  we will use the coordinate system  $\varphi : \mathbb{R}^2 \times \mathbb{R} \rightarrow P(\mathbb{R}^2)$  given by

$$\varphi(z, v) = (z, [(1, v)]).$$

It is easy to see that

$$\varphi^{-1}(z, [(v_1, v_2)]) = (z, v_2/v_1).$$

In these coordinates  $(p, E_p^u)$  is given by  $(0, 0)$ . Now we compute the linearization of  $Pf$  around  $(0, 0)$ .

We have

$$Pf(z, v) = (f(z), [Df(z) \cdot (1, v)^T]) = (Df(0) \cdot z + o(|z|), [Df(z) \cdot (1, v)^T]).$$

Observe that

$$\begin{aligned} Df(z) \cdot (1, v)^T &= \begin{pmatrix} \frac{\partial f_1}{\partial x}(z) & \frac{\partial f_1}{\partial y}(z) \\ \frac{\partial f_2}{\partial x}(z) & \frac{\partial f_2}{\partial y}(z) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ v \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ v \end{pmatrix} + \\ &\begin{pmatrix} \frac{\partial^2 f_1}{\partial x^2}(0)x + \frac{\partial^2 f_1}{\partial x \partial y}(0)y & \frac{\partial^2 f_1}{\partial y \partial x}(0)x + \frac{\partial^2 f_1}{\partial y^2}(0)y \\ \frac{\partial^2 f_2}{\partial x^2}(0)x + \frac{\partial^2 f_2}{\partial x \partial y}(0)y & \frac{\partial^2 f_2}{\partial y \partial x}(0)x + \frac{\partial^2 f_2}{\partial y^2}(0)y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ v \end{pmatrix} + o(z) \cdot \begin{pmatrix} 1 \\ v \end{pmatrix} = \\ &\begin{pmatrix} \lambda + O(|(z, v)|) \\ \mu v + \frac{\partial^2 f_2}{\partial x^2}(0)x + \frac{\partial^2 f_2}{\partial x \partial y}(0)y + o(|(z, v)|) \end{pmatrix}. \end{aligned}$$

Now we have to divide the second component of the above vector by the first one. Therefore

$$\begin{aligned} v_2/v_1 &= \left( \mu v + \frac{\partial^2 f_2}{\partial x^2}(0)x + \frac{\partial^2 f_2}{\partial x \partial y}(0)y + o(|(z, v)|) \right) \cdot \lambda^{-1} (1 + O(|(z, v)|)) = \\ &\frac{\mu}{\lambda} v + \frac{1}{\lambda} \frac{\partial^2 f_2}{\partial x^2}(0)x + \frac{1}{\lambda} \frac{\partial^2 f_2}{\partial x \partial y}(0)y + o(|(z, v)|). \end{aligned}$$

We have proved that the linearization of  $Pf$  at  $p = (0, 0)$  has the following form

$$DPf(p, E_p^u) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ \frac{1}{\lambda} \frac{\partial^2 f_2}{\partial x^2}(0) & \frac{1}{\lambda} \frac{\partial^2 f_2}{\partial x \partial y}(0) & \frac{\mu}{\lambda} \end{pmatrix}.$$

Hence we see that  $(p, E_p^u)$  is a hyperbolic fixed point for  $Pf$  with one-dimensional unstable and two-dimensional stable manifolds.

Let us consider now  $(p, E_p^s)$ . This time we will use the coordinate system  $\varphi : \mathbb{R}^2 \times \mathbb{R} \rightarrow P(\mathbb{R}^2)$  given by

$$\varphi(z, v) = (z, [(v, 1)]).$$

It is easy to see that

$$\varphi^{-1}(z, [(v_1, v_2)]) = (z, v_1/v_2).$$

Similar computations lead to the following formula for the linearization of  $Pf$  at  $(p, E_p^s)$

$$DPf(p, E_p^s) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ \frac{1}{\mu} \frac{\partial^2 f_1}{\partial x \partial y}(0) & \frac{1}{\mu} \frac{\partial^2 f_1}{\partial y^2}(0) & \frac{\lambda}{\mu} \end{pmatrix}.$$

■

From Theorem 1 we obtain the following

**Remark 2** *If  $f$  is as in Theorem 1 and  $W^s(p, f)$  and  $W^u(p, f)$  have a nonempty intersection, then*

- *if  $W^s(p, f)$  and  $W^u(p, f)$  are tangent, then we obtain a heteroclinic connection from  $(p, E_p^u)$  to  $(p, E_p^s)$ ,*
- *if  $W^s(p, f)$  and  $W^u(p, f)$  intersect transversally, then we obtain a homoclinic connection from  $(p, E_p^u)$  to  $(p, E_p^u)$ .*

### 3 Generic unfolding of quadratic tangency as the transversality question

We assume that we have two curves in  $\mathbb{R}^2$  depending on some parameter  $a$  and given by  $u(a, t) = (f_1(a, t), f_2(a, t))$  and  $s(a, t) = (g_1(a, t), g_2(a, t))$ . We are interested in establishing conditions, which will imply the existence of the generic unfolding of the quadratic tangency between them. Our goal is to formulate such conditions as the transversality question. This is Theorem 2.1 from [AM], where it was stated without proof.

**Definition 4** [PT, Sec. 3.1] *Let  $I, J, Z \subset \mathbb{R}$  be intervals. Let  $u_\mu: I \rightarrow \mathbb{R}^2$  and  $s_\mu: J \rightarrow \mathbb{R}^2$  for  $\mu \in Z$  be two smooth curves depending on  $\mu$  in the smooth way, such that  $u_{\mu_0}(t_u) = s_{\mu_0}(t_s) = q_0$  and  $u$  and  $s$  are tangent at  $q_0$ .*

*Assume there exists a  $\mu$ -dependent coordinates in a neighborhood of  $q$  for  $\mu$  close to  $\mu_0$ , such that in these coordinates we can use  $x_1$  (the first coordinate) as the parameter of our curves and the following holds*

$$\begin{aligned} s_\mu(x_1) &= (x_1, 0), \\ u_\mu(x_1) &= (x_1, ax_1^2 + b(\mu - \mu_0)) \end{aligned} \tag{2}$$

*where  $a \neq 0, b \neq 0$ . Then we say that the quadratic tangency of  $u$  and  $s$  unfolds generically.*

**Remark 3** *It is easy to see that in the above definition we can exchange the role of curves  $u$  and  $s$  using the coordinate transformation given by  $\phi_\mu(x_1, x_2) = (x_1, x_2 - ax_1^2 - b(\mu - \mu_0))$ .*

**Remark 4** In the context of Def. 4, if  $u_\mu(x_1) = g(\mu, x_1)$  then instead of (2) it is enough to require

$$\begin{aligned} g(\mu_0, 0) &= 0, & \frac{\partial g}{\partial x_1}(\mu_0, 0) &= 0, \\ \frac{\partial g}{\partial \mu}(\mu_0, 0) &\neq 0, & \frac{\partial^2 g}{\partial x_1^2}(\mu_0, 0) &\neq 0. \end{aligned}$$

**Proof:** Observe that from our assumptions follows that

$$u_\mu(x_1) = (x_1, ax_1^2 + b(\mu - \mu_0) + \Delta(\mu, x_1)),$$

where

$$\Delta(\mu_0, 0) = 0, \quad \frac{\partial \Delta}{\partial \mu}(\mu_0, 0) = 0, \quad \frac{\partial \Delta}{\partial x_1}(\mu_0, 0) = 0, \quad a \neq 0, \quad b \neq 0,$$

Observe that through the parameter dependent coordinate chart  $\phi_\mu(x_1, x_2) = (x_1, x_2 - \Delta(\mu, x_1))$  we obtain (2). ■

**Theorem 5** Let  $\Lambda \subset \mathbb{R}$  be an interval. Let  $u : \Lambda \times \mathbb{R} \rightarrow \mathbb{R}^2$  and  $s : \Lambda \times \mathbb{R} \rightarrow \mathbb{R}^2$  be two  $C^2$ -curves depending on parameter  $a \in \Lambda$ . Let  $a_0 \in \Lambda$  be a parameter at which curves  $u_a$  and  $s_a$  are tangent.

These curves have a quadratic tangency at  $a_0$  which unfolds generically iff there exists  $t_u, t_s \in \mathbb{R}$ , such that surfaces

$$E(u), E(s) : \Lambda \times \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{RP}^1$$

given by

$$\begin{aligned} E(u)(a, t) &= \left( a, u(a, t), \left[ \frac{\partial u}{\partial t}(a, t) \right] \right) \\ E(s)(a, t) &= \left( a, s(a, t), \left[ \frac{\partial s}{\partial t}(a, t) \right] \right) \end{aligned}$$

intersect transversally at the point  $E(u)(a_0, t_u) = E(s)(a_0, t_s)$ .

Before the proof we need one lemma.

**Lemma 6** Let  $s = (s_1, s_2) : \Lambda \times \mathbb{R} \rightarrow \mathbb{R}^2$  be a  $C^2$ -map, such that  $\frac{\partial s_1}{\partial t}(a_0, t_0) \neq 0$  for some  $(a_0, t_0)$ .

Then there exists a neighborhood  $\Lambda'$  of  $a_0$ , an open set  $V$ , such that  $s(a_0, t_0) \in V$ , and  $a$ -dependent coordinates on  $V$  for  $a \in \Lambda'$ , i.e.  $\phi_a : V \rightarrow \mathbb{R}^2$  for  $a \in \Lambda'$ , such that after a suitable reparameterization in these new coordinates the mapping  $s$  has locally the following form

$$s(a, t) = (t, 0).$$

**Proof:** Let us denote by  $(x, y)$  coordinates in  $\mathbb{R}^2$ . By taking a suitable parameterization and shifting the coordinates origin and permuting, if necessary, the coordinates in  $\mathbb{R}^2$  we can assume that  $t_0 = 0$ ,  $s(a_0, 0) = 0$  and  $\frac{\partial s_1}{\partial t}(a_0, 0) > 0$ . We can locally (in a suitable open set  $U$ ) use  $x$  as the parameter of a curve  $s(a, \cdot)$  for  $a \in \tilde{\Lambda} \subset \Lambda$ . Therefore, we have

$$s(a, t) = (t, s_2(a, t)).$$

Consider the following map  $\varphi : \tilde{\Lambda} \times U \rightarrow \tilde{\Lambda} \times \mathbb{R}^2$  given by

$$\varphi(a, x, y) = (a, x, y - s_2(a, x)).$$

We have

$$D\varphi(a, x, y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\partial s_2}{\partial a}(a, x) & -\frac{\partial s_2}{\partial x}(a, x) & 1 \end{bmatrix}$$

Therefore,  $\varphi$  is a local diffeomorphism. Let open sets  $\Lambda', V$  be such that  $\varphi : \Lambda' \times V \rightarrow \Lambda' \times \mathbb{R}^2$  is a diffeomorphism on the image and  $(a_0, 0) \in \Lambda' \times V$ . The  $a$ -dependent coordinates on  $V$  are given by  $\phi_a(x, y) = \varphi(a, x, y)$ .

It is easy to see that in the new coordinates given by  $\phi_a$  the curve  $s$  has the following form

$$s(a, t) = (t, 0).$$

■

**Proof of Theorem 5:** By Lemma 6 we can assume that the curve  $s$  is given by

$$s(a, t) = (t, 0).$$

Observe that the transversality implies that in the neighborhood of the intersection point the curve  $u$  can be represented as follows

$$u(a, t) = (t, g_2(a, t)),$$

where  $g_2$  satisfies the following conditions

$$g_2(a_0, 0) = 0, \quad \frac{\partial g_2}{\partial t}(a_0, 0) = 0. \quad (3)$$

Now we will prove that the transversality of  $E(u)$  and  $E(s)$  is equivalent to

$$\begin{aligned} \frac{\partial g_2}{\partial a}(a_0, 0) &\neq 0, \\ \frac{\partial^2 g_2}{\partial t^2}(a_0, 0) &\neq 0. \end{aligned}$$

Observe that from Remark 4 it follows that the above conditions together with (3) are equivalent to the generic unfolding of the quadratic tangency.

Observe that  $\frac{\partial s}{\partial t}(a, t) = (1, 0)$  and  $\frac{\partial u}{\partial t}(a_0, t = 0) = (1, 0)$  therefore in the neighborhood of  $[\frac{\partial s}{\partial t}(a, t)]$  we can use the second coordinate as a chart map in  $\mathbb{RP}^1$ .

In these coordinates we have

$$\begin{aligned} E(s)(a, t) &= (a, t, 0, 0)^T, \\ E(u)(a, t) &= (a, t, g_2(a, t), \frac{\partial g_2}{\partial t}(a, t))^T. \end{aligned}$$

We have

$$T_{E(s)(a_0, t=0)} = \text{span} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$T_{E(u)(a_0, t=0)} = \text{span} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial g_2}{\partial a}(a_0, 0) & \frac{\partial g_2}{\partial t}(a_0, 0) \\ \frac{\partial^2 g_2}{\partial t \partial a}(a_0, 0) & \frac{\partial^2 g_2}{\partial t^2}(a_0, 0) \end{pmatrix}.$$

From (3) it follows that the transversality question is equivalent to the following determinant being nonzero

$$\det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{\partial g_2}{\partial a}(a_0, 0) & 0 \\ 0 & 0 & \frac{\partial^2 g_2}{\partial t \partial a}(a_0, 0) & \frac{\partial^2 g_2}{\partial t^2}(a_0, 0) \end{pmatrix} = \frac{\partial g_2}{\partial a}(a_0, 0) \cdot \frac{\partial^2 g_2}{\partial t^2}(a_0, 0).$$

This finishes the proof. ■

## 4 How to prove the homoclinic tangency using the covering relations and the cone conditions?

We assume that the reader is familiar with the following notions: h-sets, covering relations, cone conditions and horizontal and vertical disks as defined in [KWZ, Z].

We consider a planar map  $f_a : \mathbb{R}^2 \supset \text{dom}(f) \rightarrow \mathbb{R}^2$  depending on the parameter  $a$ , which has a hyperbolic fixed point  $p_a$  with one-dimensional unstable and stable manifolds. Hence according to the setting from Section 2 we will work in four-dimensional space, using coordinates  $(a, x, y, v)$ , where  $a$  is the parameter,  $(x, y) \in \mathbb{R}^2$  and  $v$  represents points in  $P(T_{(x, y)}\mathbb{R}^2)$ , which we will call the *tangential* coordinate. We have the map (we abuse the notation for  $Pf$ )

$$Pf(a, x, y, v) = (a, Pf(x, y, v)).$$

In our method we need the following ingredients

- the chain of covering relations

$$N_0 \xrightarrow{Pf} N_1 \xrightarrow{Pf} \dots \xrightarrow{Pf} N_k, \quad (4)$$

such that the cone conditions are satisfied,

- $(a, W^u((p_a, E_{p_a}^u), Pf_a))$  as the horizontal disk in  $N_0$  satisfying the cone conditions,
- $(a, W^s((p_a, E_{p_a}^s), Pf_a))$  as the vertical disk in  $N_k$  satisfying the cone conditions.

If the above conditions are satisfied then from [Z, Theorem 7] it follows that  $(a, W^u((p_a, E_{p_a}^u), Pf_a))$  contains a horizontal disk satisfying the cone conditions in  $N_k$ . Hence  $(a, W^u((p_a, E_{p_a}^u), Pf_a))$  and  $(a, W^s((p_a, E_{p_a}^s), Pf_a))$  intersect transversally in  $N_k$ , which by Theorem 5 implies that sets  $W^u((p_a, E_{p_a}^u), Pf_a)$  and  $W^s((p_a, E_{p_a}^s), Pf_a)$  have a quadratic tangency which unfolds generically.

Since the parameter  $a$  is not changing under  $Pf$ , apparently there is a problem with the realization of the above scenario, because the covering relations together with the cone conditions imply the hyperbolicity. The essential point is that this remark is valid for closed loops of covering relations, but in our setting we just want a chain of covering relations going from one part of our phase space to another one. While constructing such chain we will arbitrarily decide whether we treat the parameter as the "unstable"/"stable" direction, by simply adjusting the sizes of h-sets in  $a$ -direction.

To make our scheme to work we need:

- to set the dimensions  $u, s$  in our h-sets  $N_i$  to be equal to 2, because this is the dimension of  $(a, W^u((p_a, E_{p_a}^u), Pf_a))$  and  $(a, W^s((p_a, E_{p_a}^s), Pf_a))$ ,
- $(a, W^u((p_a, E_{p_a}^u), Pf_a))$  should be a horizontal disk in  $N_0$ , hence we should treat  $a$  as one of unstable directions,
- $(a, W^s((p_a, E_{p_a}^s), Pf_a))$  should be a vertical disk in  $N_k$ , hence we should treat  $a$  as one of stable directions.

#### 4.1 The Lipschitz dependence of stable and unstable manifolds on parameters

We need prove that  $(a, W^u((p, E_p^u), Pf_a))$  is a horizontal disk in a suitable h-set  $N_0$ , this requires at least the Lipschitz dependence on  $a$  of the invariant manifold. In [Z, Sec. 8.2] the Lipschitz dependence of (un)stable manifolds with respect to parameters with explicit and computable constants was discussed with the eye toward the computer assisted proofs. Here we will just recall (and refine a bit some estimates) these results in a form of the cone conditions.

We will be using the norms for quadratic forms (identified in the sequel with symmetric matrices) which are defined by

$$|B(u, v)| \leq \|B\| \|u\| \|v\|.$$

For Euclidian norm we have

$$\|B\| = \max\{|s| \mid s \text{ is an eigenvalue of } B\}.$$

**Theorem 7** Assume that  $(N, Q)$  is an  $h$ -set in  $\mathbb{R}^{u+s}$  with cones and  $f_\lambda: \mathbb{R}^{u+s} \rightarrow \mathbb{R}^{u+s}$  with  $\lambda \in C$ , where  $C$  is a compact interval in the parameter space and  $Q$  has the form  $Q(x, y) = \alpha(x) - \beta(y) = \sum_{i=1}^u a_i x_i^2 - \sum_{i=1}^s a_{i+u} y_i^2$ .

1. Assume that for the covering relation  $N \xrightarrow{f_\lambda} N$  the cone condition is satisfied for all  $\lambda \in C$ .
2. Let  $\epsilon > 0$  and  $A > 0$  be such that for all  $\lambda \in C$  and  $z_1, z_2 \in N$  holds

$$Q(f_\lambda(z_1) - f_\lambda(z_2)) - (1 + \epsilon)Q(z_1 - z_2) \geq A(z_1 - z_2)^2. \quad (5)$$

3. Let

$$M = \max_{\lambda \in C, z \in N} \left( \sum_i |a_i| \left\| \frac{\partial \pi_{z_i} f_\lambda}{\partial z}(z) \right\| \cdot \left\| \frac{\partial \pi_{z_i} f_\lambda}{\partial \lambda}(z) \right\| \right), \quad (6)$$

$$L = \|\beta\| \cdot \max_{\lambda \in C, z \in N} \left\| \frac{\partial \pi_y f_\lambda}{\partial \lambda}(z) \right\|^2. \quad (7)$$

4. Let  $\Gamma > 0$  be such that

$$A - 2M\Gamma - L\Gamma^2 > 0. \quad (8)$$

5. We define

$$\delta = \frac{\Gamma^2}{\|\alpha\|}. \quad (9)$$

Then the set  $W_N^s(p_\lambda, f_\lambda)$  for  $\lambda \in C$  can be parameterized as a vertical disk in  $C \times N$  for the quadratic form  $\tilde{Q}(\lambda, z) = \delta Q(z) - \lambda^2$ .

Before the proof let us make two observations concerning constants  $A, \epsilon, \Gamma$ .

**Remark 8** The existence of  $A$  and  $\epsilon$  in (5) is a consequence of the cone condition for covering relation  $N \xrightarrow{f_\lambda} N$ . We would like to have as big  $A$  as possible. This forces  $\epsilon \rightarrow 0$ , but  $\epsilon$  is not used in the sequel.

**Remark 9** Since  $A > 0$ , therefore  $\Gamma$  in (8) always exists, but it is desirable to look for largest  $\Gamma$  possible.

**Proof of Theorem 7:** We would like to obtain that for  $|\lambda_1 - \lambda_2| \leq \Gamma \|z_1 - z_2\|$  holds

$$Q(f_{\lambda_1}(z_1) - f_{\lambda_2}(z_2)) > (1 + \epsilon)Q(z_1 - z_2).$$

Let  $B$  be a unique symmetric form, such that  $B(u, u) = Q(u)$ . Observe that

$$\begin{aligned} & Q(f_{\lambda_1}(z_1) - f_{\lambda_2}(z_2)) - (1 + \epsilon)Q(z_1 - z_2) = \\ & Q(f_{\lambda_1}(z_1) - f_{\lambda_1}(z_2)) - (1 + \epsilon)Q(z_1 - z_2) + \\ & 2B(f_{\lambda_1}(z_1) - f_{\lambda_1}(z_2), f_{\lambda_1}(z_2) - f_{\lambda_2}(z_2)) + Q(f_{\lambda_1}(z_2) - f_{\lambda_2}(z_2)). \end{aligned}$$

The first term in the above expression will be estimated using (5).

For the third term we obtain

$$\begin{aligned} & Q(f_{\lambda_1}(z_2) - f_{\lambda_2}(z_2)) \geq -\beta(\pi_y(f_{\lambda_1}(z_2) - f_{\lambda_2}(z_2))) \geq \\ & -\|\beta\| \cdot \max_{\lambda \in C} \left\| \frac{\partial \pi_y f_\lambda}{\partial \lambda}(z_2) \right\|^2 \cdot (\lambda_1 - \lambda_2)^2 \geq \\ & -\|\beta\| \max_{\lambda \in C, z \in N} \left\| \frac{\partial \pi_y f_\lambda}{\partial \lambda}(z) \right\|^2 \cdot \Gamma^2 \|z_1 - z_2\|^2 = -L\Gamma^2 \|z_1 - z_2\|^2. \end{aligned}$$

Finally, for the second term we have

$$\begin{aligned} & |B(f_{\lambda_1}(z_1) - f_{\lambda_1}(z_2), f_{\lambda_1}(z_2) - f_{\lambda_2}(z_2))| \leq \\ & \max_{\lambda \in C, z \in N} \left( \sum_i |a_i| \left\| \frac{\partial \pi_{z_i} f_\lambda}{\partial z}(z) \right\| \cdot \left\| \frac{\partial \pi_{z_i} f_\lambda}{\partial \lambda}(z) \right\| \right) \cdot \Gamma \|z_1 - z_2\|^2. \end{aligned}$$

From the above computations and (6–7) we obtain the following

$$Q(f_{\lambda_1}(z_1) - f_{\lambda_2}(z_2)) - (1 + \epsilon)Q(z_1 - z_2) \geq (A - 2M\Gamma - L\Gamma^2) \|z_1 - z_2\|^2.$$

For  $\Gamma$  and  $\delta$  as in (8) and (9) it is proved in [Z, Lemma 23] that, for  $\lambda_1, \lambda_2 \in C$ ,  $\lambda_1 \neq \lambda_2$  and  $z_i \in W_N^s(p_{\lambda_i}, f_{\lambda_i})$  for  $i = 1, 2$  holds

$$\delta Q(z_1 - z_2) - (\lambda_1 - \lambda_2)^2 < 0.$$

Hence  $W_N^s(p_\lambda, f_\lambda)$  for  $\lambda \in C$  is vertical disk in  $C \times N$  for the quadratic form  $\tilde{Q}(\lambda, z) = \delta Q(z) - \lambda^2$ . ■

#### Comments:

- Since  $\tilde{Q}(x, y, \lambda) = \delta\alpha(x) - \delta\beta(y) - \lambda^2$ , hence due to the negative sign in front of  $\lambda^2$  it follows that when trying to represent the stable manifold as the vertical disk in  $C \times N$ , we must treat the parameter as a 'stable' direction in an h-set.
- For the unstable manifold we have to take the inverse map and we obtain different  $\delta$ . Since taking the inverse involves changing the sign of  $Q$  we end up with the following quadratic form

$$\tilde{Q}(x, y, \lambda) = \delta\alpha(x) + \lambda^2 - \delta\beta(y).$$

Looking at sign in front of  $\lambda^2$  we see that  $\lambda$  appears as an 'unstable' direction in  $C \times N$ .

## 5 A toy example

In this section we show how to construct the chain of covering relations (4) discussed in the first part of Section 4 for a special model map with a quadratic tangency which unfolds generically. Our intention is that the reader may see and appreciate some details of the method, which are later hardly visible when we report on the computer assisted proofs for the Hénon map and the forced pendulum in Sections 6 and 7, respectively.

We define a map  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  depending on the parameter  $a$  as follows:

- $(0,0)$  is a hyperbolic fixed point and in a neighborhood of  $(0,0)$  the map  $f_a$  is linear

$$f_a(x, y) = (\lambda x, \mu y)$$

where  $|\lambda| > 1$  and  $|\mu| < 1$ ,

- in a neighborhood of the point  $(1, 0)$  we have the homoclinic tangency for  $a = 0$ . We assume that  $f_a$  acts as follows

$$f_a(1 + x, y) = (x^2 + y + a, 1 - x).$$

When compared with the full problem of establishing the existence of generic unfolding of homoclinic tangency for the Hénon map and the forced damped pendulum, the above model map avoids the problems related to providing the explicit estimates on the dependence of  $W^{s,u}$  on the parameter, which are given in Theorem 7.

### 5.1 Chain of covering relations

Let  $p = (0, 0)$ , then  $E_p^u = [(1, 0)]$  and  $E_p^s = [(0, 1)]$ . Our goal is to construct a chain of covering relations 'linking'  $(p, E_p^u)$  with  $(p, E_p^s)$ .

#### The beginning of the chain:

Let us first see how the covering relations look in a neighborhood of  $(p, E_p^u)$ . We use the chart  $(x, y, v, a)$ , where  $v \mapsto [(1, v)]$ . Therefore the map  $F = (Pf_a, a)$  works as follows

$$F(x, y, v, a) = (\lambda x, \mu y, (\mu/\lambda)v, a).$$

We define an  $h$ -set, so that  $(x, a)$  are the 'unstable' directions,  $(y, v)$  are the 'stable' ones. It is easy to see, that there exists a sequence of  $h$ -sets  $N_i = (c_i + [-x_i, x_i]) \times [-y_i, y_i] \times [-v_i, v_i] \times [-a_i, a_i]$ , with  $c_0 = 0$ ,  $c_k = 1$  and such that

$$N_0 \xrightarrow{F} N_1 \xrightarrow{F} \dots \xrightarrow{F} N_k.$$

Observe that the necessary conditions are

$$\begin{aligned} c_{i+1} + [-x_{i+1}, x_{i+1}] &\subset \lambda c_i + [-|\lambda|x_i, |\lambda|x_i], \\ |\mu|y_i &< y_{i+1}, \\ |\mu/\lambda|v_i &< v_{i+1}, \\ a_i &> a_{i+1}. \end{aligned}$$

**The end of the chain:**

In a neighborhood of  $(p, E_p^s)$  the 'unstable' subspace is  $(x, w)$ , where  $w$  corresponds to the  $[(w, 1)]$ . The map  $F$  works as follows

$$F(x, y, w, a) = (\lambda x, \mu y, (\lambda/\mu)w, a).$$

It is easy to see, that there exists a sequence of  $h$ -sets  $M_i = (\bar{c}_i + [-\bar{x}_i, \bar{x}_i]) \times [-\bar{y}_i, \bar{y}_i] \times [-\bar{w}_i, \bar{w}_i] \times [-\bar{a}_i, \bar{a}_i]$ , such that

$$\bar{c}_0 = 0, \quad \bar{c}_s = 1, \quad M_s \xrightarrow{F} M_{s-1} \xrightarrow{F} \dots \xrightarrow{F} M_0.$$

Observe that the necessary conditions are

$$\begin{aligned} (\bar{c}_{i+1} + [-\bar{x}_{i+1}, \bar{x}_{i+1}]) &\subset \mu \bar{c}_i + [-|\mu| \bar{x}_i, |\mu| \bar{x}_i], \\ \bar{y}_{i+1} &> |\mu| \bar{y}_i, \\ \bar{w}_{i+1} &< |\lambda/\mu| \bar{w}_i, \\ \bar{a}_{i+1} &< \bar{a}_i. \end{aligned}$$

**Switching from unstable to the stable manifold.** We want the covering relation  $N_k \xrightarrow{F} M_s$ , where  $N_k$  and  $M_s$  are as above.

In  $N_k$  the nominally unstable directions are  $(x, a)$ , while in  $M_s$  parameterized by  $(x, y, w, a)$ , where  $w \mapsto [(w, 1)]$  the 'unstable' direction is  $(x, w)$ . We have

$$F(1+x, y, v, a) = (x^2 + y + a, 1-x, -(2x+v), a).$$

We would like to have the following homotopy for covering relation

$$G_t(1+x, y, v, a) = (a + t(x^2 + y), 1-tx, -2x-tv, ta).$$

Therefore if we want a covering relation  $N_k \xrightarrow{F} M_s$ , where  $N_k = (1, 0, 0, 0) + [-x_k, x_k] \times [-y_k, y_k] \times [-v_k, v_k] \times [-a_k, a_k]$  and  $M_s = (0, 1, 0, 0) + [-\bar{x}_s, \bar{x}_s] \times [-\bar{y}_s, \bar{y}_s] \times [-\bar{w}_s, \bar{w}_s] \times [-\bar{a}_s, \bar{a}_s]$ , then we need to satisfy the following set of inequalities

$$\begin{aligned} a_k - x_k^2 - y_k &> \bar{x}_s, \\ 2x_k - v_k &> \bar{w}_s, \\ \bar{y}_s &> x_k, \\ \bar{a}_s &> a_k. \end{aligned}$$

It is easy to see that this set of inequalities has a solution. For example if  $a_k = \Delta < 1$ , then we can choose (for some small  $\epsilon > 0$ )

$$\begin{aligned} x_k &= \Delta/2, \\ y_k &= \bar{x}_s = \Delta/3, \\ v_k &= \bar{w}_s = (1-\epsilon)\Delta/2, \\ \bar{y}_s &= (0.5+\epsilon)\Delta, \\ \bar{a}_s &= (1+\epsilon)\Delta. \end{aligned}$$

Observe that the expanding direction  $x$  is stretched in  $w$ -direction and  $a$  direction is stretched across  $x$ -direction (in the target set).

It is clear that we can easily build the desired chain of covering relation of the form

$$N_0 \xrightarrow{F} N_1 \xrightarrow{F} \dots \xrightarrow{F} N_k \xrightarrow{F} M_s \xrightarrow{F} M_{s-1} \xrightarrow{F} \dots \xrightarrow{F} M_0.$$

## 5.2 Cone conditions

It turns out that the cone conditions at the beginning and the end of the chain (4) are relatively easy to satisfy in the situation when dynamics is linear (there is no parameter dependence) in a neighborhood of the hyperbolic fixed point, otherwise the issue becomes delicate see Theorem 7 in Section 4.1.

As a rule in this subsection the coordinate order for h-sets  $N, M$  will such that always the nominally unstable coordinates are written first.

**At the beginning of the chain** For covering relations  $N_i \xrightarrow{F} N_{i+1}$  the expanding directions are  $(x, a)$ . Let  $N_i$  be an h-set with  $Q_i$ -cones given by

$$Q_{N_i}(x, a, y, v) = \alpha_i x^2 + \beta_i a^2 - \gamma_i y^2 - \delta_i v^2.$$

The map  $F$  is linear for  $N_i \xrightarrow{F} N_{i+1}$  therefore it is enough to check whether

$$Q_{N_{i+1}}(F(x, a, y, v)) > Q_{N_i}(x, a, y, v).$$

We have

$$\begin{aligned} Q_{N_{i+1}}(F(x, a, y, v)) - Q_{N_i}(x, a, y, v) &= \\ \alpha_{i+1} \lambda^2 x^2 + \beta_{i+1} a^2 - \gamma_{i+1} \mu^2 y^2 - (\mu/\lambda)^2 \delta_{i+1} v^2 - \\ &\quad (\alpha_i x^2 + \beta_i a^2 - \gamma_i y^2 - \delta_i v^2) = \\ (\alpha_{i+1} \lambda^2 - \alpha_i) x^2 + (\beta_{i+1} - \beta_i) a^2 + (\gamma_i - \mu^2 \gamma_{i+1}) y^2 + (\delta_i - (\mu/\lambda)^2 \delta_{i+1}) v^2 \end{aligned}$$

Hence we will have cone conditions satisfied, when for example  $\alpha_{i+1} = \alpha_i$ ,  $\gamma_{i+1} = \gamma_i$ ,  $\delta_{i+1} = \delta_i$ . The only strict requirement is

$$\beta_{i+1} > \beta_i.$$

**At the end of the chain.** It goes through as easily as the previous case. For covering relations  $M_i \xrightarrow{F} M_{i-1}$  the unstable directions are  $(x, w)$ . We set

$$Q_{M_i}(x, w, y, a) = A_i x^2 + B_i w^2 - C_i y^2 - D_i a^2.$$

As above it is enough to whether

$$Q_{M_{i-1}}(F(x, w, y, a)) > Q_{M_i}(x, w, y, a).$$

We have

$$\begin{aligned} Q_{M_{i-1}}(F(x, w, y, a)) - Q_{M_i}(x, w, y, a) &= \\ (A_{i-1} \lambda^2 - A_i) x^2 + (B_{i-1} (\lambda/\mu)^2 - B_i) w^2 + \\ &\quad (C_{i-1} - C_i \mu^2) y^2 + (D_{i-1} - D_i) a^2. \end{aligned}$$

We see that we need to have

$$D_{i-1} > D_i.$$

For the remaining coefficients we can set

$$A_i = A_{i-1}, B_{i-1} = B_i, C_{i-1} = C_i.$$

**Switching from the unstable to the stable manifold.** Consider the covering relation  $N_k \xrightarrow{F} M_s$ . This time we are in the nonlinear regime. We use on  $N_k$  coordinates  $(x, a, y, v)$  and on  $M_s$  coordinates  $(x, w, y, a)$

This means that in these coordinates (we remove the shifts of the origin of coordinate frame to  $(1, 0, 0, 0)$  and  $(0, 0, 1, 0)$  for  $N_k$  and  $M_s$  respectively)

$$F(x, a, y, v) = (x^2 + y + a, -2x - v, -x, a).$$

We set

$$\begin{aligned} Q_N(x, a, y, v) &= \alpha x^2 + \beta a^2 - \gamma y^2 - \delta v^2, \\ Q_M(x, w, y, a) &= Ax^2 + Bw^2 - Cy^2 - Da^2. \end{aligned}$$

We have

$$DF(x, a, y, v) = \begin{bmatrix} 2x & 1 & 1 & 0 \\ -2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

After some computations we obtain

$$\tilde{Q} = (DF)^T Q_M DF - Q_N = \begin{bmatrix} 4Ax^2 + (4B - C - \alpha) & 2Ax & 2Ax & 2B \\ 2Ax & A - D - \beta & A & 0 \\ 2Ax & A & A + \gamma & 0 \\ 2B & 0 & 0 & B + \delta \end{bmatrix}.$$

We are interested whether  $\tilde{Q}$  is positive definite. Observe that since the positive definiteness is an open condition we will show that  $\tilde{Q}$  is positive definite for  $x = 0$  and then we will know that the same holds for  $|x|$  small.

After we set  $x = 0$  we obtain

$$\tilde{Q} = \begin{bmatrix} 4B - C - \alpha & 0 & 0 & 2B \\ 0 & A - D - \beta & A & 0 \\ 0 & A & A + \gamma & 0 \\ 2B & 0 & 0 & B + \delta \end{bmatrix}.$$

It is easy to see that after rearrangement of coordinates the question is reduced to the positive definiteness of the following two matrices

$$\tilde{Q}_1 = \begin{bmatrix} 4B - C - \alpha & 2B \\ 2B & B + \delta \end{bmatrix}, \quad \tilde{Q}_2 = \begin{bmatrix} A - D - \beta & A \\ A & A + \gamma \end{bmatrix}.$$

For example, we can set  $A = B = C = 1$  and  $D = \frac{1}{2}$ , then we obtain

$$\tilde{Q}_1 = \begin{bmatrix} 3 - \alpha & 2 \\ 2 & 1 + \delta \end{bmatrix}, \quad \tilde{Q}_2 = \begin{bmatrix} 1/2 - \beta & 1 \\ 1 & 1 + \gamma \end{bmatrix}.$$

It is now easy to see that we get what we want, when we set

$$\begin{aligned} \alpha &= 1, & \delta &> 1, \\ \beta &= 1/4, & \gamma &> 3. \end{aligned}$$

Summarizing we obtained

$$\begin{aligned} Q_N(x, a, y, v) &= x^2 + a^2/4 - 4y^2 - 2v^2, \\ Q_M(x, w, y, a) &= x^2 + w^2 - y^2 - a^2/2. \end{aligned}$$

## 6 Application to the Hénon map.

In this section we will show that the method introduced in the previous sections can be successfully applied to a specific system. Let us consider the Hénon map

$$H_{a,b}(x, y) = (a - x^2 + by, x). \quad (10)$$

The following theorems have been proven in [AM].

**Theorem 10** [AM, Thm.1.1] *There exists an open neighborhood  $B$  of parameter value  $b = 0.3$  such that for each parameter  $b \in B$  there exists a parameter value*

$$a \in [1.392419807915, 1.392419807931]$$

*such that the Hénon map has a quadratic tangency unfolding generically for the fixed point*

$$x_{a,b} = y_{a,b} = -\frac{1}{2} \left( b - \sqrt{(b-1)^2 + 4a} - 1 \right).$$

**Theorem 11** [AM, Thm.1.2] *There exists an open neighborhood  $B$  of parameter value  $b = -0.3$  such that for each parameter  $b \in B$  there exists a parameter value*

$$a \in [1.314527109319, 1.314527109334] \quad (11)$$

*such that the Hénon map has a quadratic tangency unfolding generically for the fixed point*

$$x_{a,b} = y_{a,b} = \frac{1}{2} \left( b - \sqrt{(b-1)^2 + 4a} - 1 \right).$$

The numerical evidence of the existence of a homoclinic tangency for some parameter values  $(a, b) \approx (1.3145271093265, -0.3)$  is shown on Fig. 1 – see also [AM, Fig. 1.1]

The main motivation for us to study the existence of homoclinic tangencies for the Hénon map was to verify if our method could work in this relatively easy example.

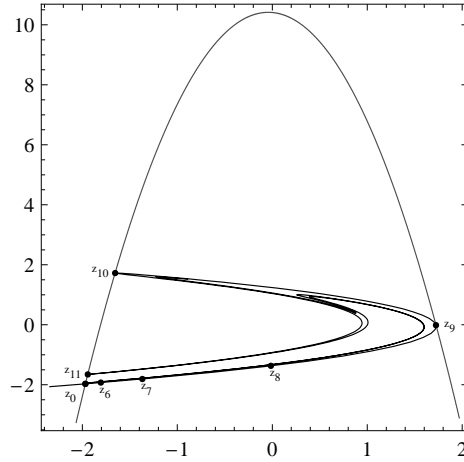


Figure 1: Parts of the unstable and stable manifolds of the Hénon map at the fixed point for  $b = -0.3$  and  $a = a_0 = 1.3145271093265$ .

Denote by

$$a_0 = 1.3145271093265$$

a center of the interval (11). The aim of this section is to prove the following theorem

**Theorem 12** *There exists an open neighborhood  $B$  of the parameter value  $b = -0.3$  such that for each  $b \in B$  there is a parameter  $a \in a_0 + [-10^{-5}, 10^{-5}]$  such that the Hénon map (10)  $H_{a,b}$  has a quadratic homoclinic tangency unfolding generically for the fixed point*

$$x_{a,b} = y_{a,b} = \frac{1}{2} \left( b - \sqrt{(b-1)^2 + 4a} - 1 \right) \approx -1.9679632427827796. \quad (12)$$

The authors in [AM] report relatively long computational time (approximately 100 minutes) for the computer assisted proof of [AM, Thm.1.2]. It turns out that using our method the verification of necessary inequalities in the computer assisted proof of Theorem 12 has been completed in 0.2 second on the Intel Xeon 5160, 3GHz processor. This very good efficiency allowed us to apply the method to a map coming from an ODE - a suitable Poincaré map in the forced-damped pendulum. The details will be given in the next section.

Computer assisted proof of Theorem 12 will be presented in the following subsections in which we verify:

- the existence of a heteroclinic chain of covering relations for  $PH$ ,
- the cone conditions along the above mentioned heteroclinic chain,

- the cone conditions at the beginning and at the end of the chain which allow us to parameterize the center-unstable and center-stable manifolds as a horizontal or vertical discs, respectively, in proper h-sets.

### 6.1 The existence of a heteroclinic chain of covering relations for $PH$ .

In Theorem 12 we chose the center of the interval (11)  $a_0 = 1.3145271093265$  as a good candidate for the homoclinic tangency parameter corresponding to  $b_0 = -0.3$ .

In order to define the sets which will appear in the heteroclinic chain, we need to set a local chart for the manifold  $P\mathbb{R}^2$ . It turns out, that it is enough for our purpose to use the parametrization

$$\psi : \mathbb{R}^2 \times (0, \pi) \ni (x, y, t) \mapsto (x, y, [(\cos(t), \sin(t))]) \in P\mathbb{R}^2. \quad (13)$$

This parametrization excludes one point on the manifold, but through our computations this point does not appear either as an argument or a value of  $PH_a$ . All the sets will be expressed using coordinates  $(x, y, t, a)$ .

Put

$$PH(x, y, t, a) = ((\psi^{-1} \circ PH_{a,b_0} \circ \psi)(x, y, t), a) \quad (14)$$

In the sequel by  $\pi_t$  we will denote a projection onto tangent coordinate, i.e.

$$\pi_t(x, y, [u]) = \frac{u}{\|u\|}.$$

With some abuse of notation we will use the same symbols for the projections

$$\pi_t(x, y, [u], a) = \frac{u}{\|u\|} \quad \text{or} \quad \pi_t(x, y, t, a) = (\cos(t), \sin(t))$$

but it is clear from the list of arguments which projection has to be used. In each case the value of  $\pi_t$  is a vector.

To simplify the notation we will use  $z_0 = (x_0, y_0) = (x_{a_0, b_0}, y_{a_0, b_0})$  as defined in (12). Since we always will have the fixed value of the parameter  $b_0 = -0.3$  we will write  $H_a$  instead of  $H_{a,b_0}$  in the sequel. Let  $u_0$  and  $s_0$  be normalized with respect to the Euclidean norm eigenvectors of  $DH_{a_0}(z_0)$  given explicitly by

$$\begin{aligned} u_0 &= \frac{(-x_0 + \sqrt{x_0^2 + b_0}, 1)}{\|(-x_0 + \sqrt{x_0^2 + b_0}, 1)\|} \approx (0.9680131177714217873, 0.250899589123719882), \\ s_0 &= \frac{(-x_0 - \sqrt{x_0^2 + b_0}, 1)}{\|(-x_0 - \sqrt{x_0^2 + b_0}, 1)\|} \approx -(0.07752307795993337433, 0.996990557820693689) \end{aligned} \quad (15)$$

and let  $M = [u_0^T, s_0^T]$  be a matrix of eigenvectors. Put

$$z_1 = z_0 + 0.0001993152279412426u_0 + 2.50404 \cdot 10^{-11}s_0.$$

The above point has been chosen as a good approximation of the homoclinic tangency point for  $H_{a_0}$ . Namely, we have

$$\begin{aligned} \|H_{a_0}^{-1}(z_1) - z_0\| &\leq 5.2 \cdot 10^{-5}, \\ \|H_{a_0}^{14}(z_1) - z_0\| &\leq 1.2 \cdot 10^{-5}, \\ M^{-1}(\pi_t(PH_{a_0}^{14}(z_1, [u_0]))) &\approx (-4.71 \cdot 10^{-7}, 0.999999847). \end{aligned} \quad (16)$$

We see that the unstable direction  $u_0$  at  $z_1$  is mapped under 14-th iterate of  $H_{a_0}$  very close to the stable direction  $s_0$  at the point  $H_{a_0}^{14}(z_1)$ . Let us underline that to get the estimation (16) we need to compute  $DH_{a_0}^{14}$  with at least **long double** precision of the floating point arithmetics.

The points on the trajectory of  $z_1$  will be the centers of the sets which appear in the computer assisted proof. Put

$$\begin{aligned} c_0 &= (\psi^{-1}(z_0, u_0), a_0), \\ c_1 &= (\psi^{-1}(z_1, u_0), a_0), \\ c_{i+1} &= PH^i(c_1), \quad \text{for } i = 1, \dots, 13, \\ c_{15} &= (\psi^{-1}(z_0, s_0), a_0). \end{aligned}$$

For further use we set also

$$\begin{aligned} z_{i+1} &= H_{a_0}^i(z_1), \quad \text{for } i = 1, \dots, 13, \\ z_{15} &= z_0. \end{aligned}$$

Some of these points are shown in Fig. 1.

Now, we have to chose an approximate stable and unstable directions at  $c_i$ ,  $i = 0, \dots, 15$ . On each set centered at  $c_i$ , the coordinate system will be given by a matrix

$$M_i = \begin{bmatrix} (u_i)_1 & (s_i)_1 & 0 & 0 \\ (u_i)_2 & (s_i)_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (17)$$

where  $u_0, s_0$  are given by (15) and  $u_i$  and  $s_i$  are computed as follows.

- Put  $u_{15} = u_0, s_{15} = s_0, u_1 = u_0, s_1 = s_0$ .
- For  $i = 2, \dots, 8$  we set

$$u_i = \pi_t(c_i), \quad (18)$$

$$s_i = \pi_t(PH^{-1}(z_{i+1}, \pi_t(c_{i+1})^\perp)). \quad (19)$$

From numerical simulations we get, that between the points  $c_8$  and  $c_9$  the role of the tangent coordinate is changing from contracting to expanding. Therefore, the unstable direction propagates very well on these sets just by (18) for  $i = 2, \dots, 8$ . For the inverse map, the unstable directions  $u_i$  become repelling. Therefore the preimage of an orthogonal direction to  $u_{i+1}$  becomes a good enough approximation for our method of the stable direction at  $z_i$ .

$i$	$10^5 \cdot (d_i)_1$ unstable dir.	$10^5 \cdot (d_i)_2$ stable dir.	$10^5 \cdot (d_i)_3$ tangent dir.	$10^5 \cdot (d_i)_4$ parameter
0	7	1	2	$(1.01)^8$
1	1	1	2	$(1.01)^7$
2	1	1	2	$(1.01)^6$
3	1	1	2	$(1.01)^5$
4	1	1	2	$(1.01)^4$
5	1	1	2	$(1.01)^3$
6	1	1	2	$(1.01)^2$
7	1	1	2	1.01
8	1	1	2	1
9	0.5	1.25	0.25	1.01
10	0.75	1.25	0.25	$(1.01)^2$
11	1	1.25	0.25	$(1.01)^3$
12	1	1.25	0.25	$(1.01)^4$
13	1	1.25	0.25	$(1.01)^5$
14	1	1.25	0.25	$(1.01)^6$
15	1	2	0.25	$(1.01)^7$

Table 1: Diameters of the h-sets in the heteroclinic sequence for the Hénon map. The diameters in the table are scaled by the factor  $10^5$ . On h-sets  $N_0, \dots, N_8$  the expanding directions are  $u$  and  $a$  coordinates, while on  $N_9, \dots, N_{15}$  the expanding directions are  $u$  and  $t$  coordinates.

- According to a good choice of the homoclinic point (16), the tangent coordinate at  $c_i$  become a good approximation of the stable direction on sets centered at  $c_9, \dots, c_{14}$ . Therefore, we can set

$$\begin{aligned}
s_i &= \pi_t(c_i), \quad \text{for } i = 9, \dots, 14, \\
u_9 &= \pi_t(PH_{a_0}(z_8, [s_8])), \\
u_{i+1} &= \pi_t(PH_{a_0}(z_i, [u_i])), \quad \text{for } i = 9, \dots, 13.
\end{aligned}$$

There remains for us to set the sizes of the sets. We define the h-sets by

$$N_i = c_i + M_i \cdot (d_i \cdot [-1, 1]), \quad (20)$$

where the diameters  $d_i$  are listed in Table 1 and chosen from numerical experiments as well working. On h-sets  $N_0, \dots, N_8$  the expanding directions are  $u$  and  $a$  coordinates, while on  $N_9, \dots, N_{15}$  the expanding directions are  $u$  and  $t$  coordinates.

Let us comment briefly about the choice of the sizes of the sets presented in Table 1. For the sets  $N_0, \dots, N_8$  the tangent coordinate is chosen as a contracting direction. Therefore the set in this direction must be large enough (here  $2 \cdot 10^{-5}$ ) to be able to easy verify that we have contraction on this variable.

On the other hand, for the sets  $N_9, \dots, N_{15}$  we have a strong expansion on this coordinate. Therefore the diameter of these sets on this coordinate is smaller here because it is easier to verify that the image of some walls of the previous set in the chain is outside of a small set than inside of it.

Moreover, the tangent coordinate of  $N_9$  has to be covered by the unstable coordinate of  $N_8$ . This expansion is weak and forces decreasing of the size  $N_9$  on the tangent coordinate.

The unstable and stable direction are comparable for almost all sets in the chain except  $N_0$ ,  $N_9$  and  $N_{15}$ . For  $N_0$  we have large size in unstable direction. This is due to the fact that  $N_1$  is in some distance from  $z_0$ . The size  $7 \cdot 10^{-5}$  is necessary to reach the set  $N_1$  from  $N_0$ .

Similar reasoning apply to  $N_{15}$ . The stable size must be large enough so that the image of  $N_{14}$  is captured in this direction.

On the set  $N_9$  we have change of dynamics. The unstable size of  $N_9$  is smaller since on  $N_8$  the parameter plays a role of expanding direction which covers unstable direction on  $N_9$ . Since this expansion is weak, we must decrease the size of  $N_9$ . On the other hand the image of  $N_8$  vary with the parameter and it is larger in the stable direction of  $N_9$ . This forces to enlarge a little bit the set  $N_9$  in the stable direction.

Now we can state the first numerical lemma.

**Lemma 13** *The following covering relations hold true*

$$N_0 \xrightarrow{PH} N_1 \xrightarrow{PH} \dots \xrightarrow{PH} N_{15}.$$

**Proof:** The assertion of the above lemma has been verified using the interval arithmetics [M] and the algorithms for verifying the existence of covering relations described in [WZ]. We were able to verify the necessary inequalities on each wall of  $N_i$ ,  $i = 0, \dots, 15$  without any subdivision of the sets, so the computational time was less than one second. ■

## 6.2 The cone conditions along the heteroclinic chain

In this section we will show that the cone conditions are satisfied for the sequence of covering relations from Lemma 13. We have two results which give us a numerical method for verifying the cone conditions.

**Lemma 14** [KWZ, Lemma 6] *Let  $(N, Q_N)$  and  $(M, Q_M)$  be  $h$ -sets with cones in  $\mathbb{R}^n$  and let  $f : N \rightarrow \mathbb{R}^n$  be  $C^1$  such that  $N \xrightarrow{f} M$ . Let  $[Df(N)]_I$  denote the interval enclosure of the set of matrices  $Df(N)$ . If the interval matrix*

$$V = [Df(N)]_I^T Q_M [Df(N)]_I - Q_N$$

*is positive definite then the cone conditions are satisfied for the covering relation  $N \xrightarrow{f} M$ .*

Let  $A = A_c + [-1, 1] \cdot A_0$  be an interval matrix, where  $A_c, A_0 \in \mathbb{R}^{n \times n}$  are real and symmetric. For  $z \in \mathbb{R}^n$  by  $\Delta(z)$  we will denote a diagonal matrix with  $z_i$ 's at the diagonal.

**Lemma 15** [*R, Theorem 2*] *The interval matrix  $A = A_c + [-1, 1] \cdot A_0$  is positive definite if and only if for each sequence  $z \in \{-1, 1\}^n$  the matrix*

$$A_z = A_c - \Delta(z)A_0\Delta(z)$$

*is positive definite.*

In the light of the above lemma to verify that a symmetric interval matrix is positive definite it is enough to verify if  $2^{n-1}$  real matrices  $A_z$  are positive definite.

In order to verify the cone conditions we have to define quadratic forms on the sets  $N_i$ ,  $i = 0, \dots, 15$ . Denote by

$$\lambda = 3.858169402, \quad \mu = 0.07775708341 \quad (21)$$

an approximate eigenvalues of  $DH_{a_0}$  at  $z_0$ . Recall, by  $\Delta(p_1, \dots, p_n)$  we denote a diagonal matrix with  $p_i$ 's at the diagonal. For  $i = 0, \dots, 15$  we define the quadratic form on the h-set  $N_i$  by

$$Q_i = \Delta((p_i)_1, (p_i)_2, (p_i)_3, (p_i)_4)$$

where the coefficients are listed in Table 2. The quadratic forms  $Q_i$  are defined in the coordinate systems given by matrixes  $M_i$  (17) used to define the h-sets  $N_i$ ,  $i = 0, \dots, 15$ . These matrices have normalized columns. In these coordinates the sets are given by  $N_i = [(d_i)_1, \dots, (d_i)_4] \cdot [-1, 1]$ ,  $i = 0, \dots, 15$  – see (20) and Table 1. Let us comment briefly about the choice of these coefficients. Assume that we would like to verify the cone conditions for the covering relation  $N \xrightarrow{f} M$ . Assume that  $f$  is linear and in some coordinate systems on  $N$  and  $M$  it is given by  $f = \Delta(\lambda_1, \dots, \lambda_k)$ . In general case, we usually have  $Df(N)$  close to a diagonal matrix, but the arguments apply. Assume also that the quadratic forms on both sets  $N$  and  $M$  are diagonal and given by

$$\begin{aligned} Q_N &= \Delta(\alpha_1^N, \dots, \alpha_k^N), \\ Q_M &= \Delta(\alpha_1^M, \dots, \alpha_k^M). \end{aligned}$$

According to Lemma 14, the cone conditions will be satisfied if the interval matrix

$$V = [Df(N)]_I^T \cdot Q_M \cdot [Df(N)]_I - Q_N = \Delta(\lambda_1^2 \alpha_1^M - \alpha_1^N, \dots, \lambda_k^2 \alpha_k^M - \alpha_k^N)$$

is positive definite. We see, that if  $|\alpha_i^{N,M}| = 1$  and some  $|\lambda_i| \gg 1$  then the corresponding coefficient in the matrix  $V$  becomes very large while for  $|\lambda_j| \ll 1$  the corresponding coefficient in  $V$  is close to 1. In general (nonlinear) case we have some nonzero intervals off the diagonal of  $V$ . Therefore, from the

$i$	$(p_i)_1$ unstable dir.	$(p_i)_2$ stable dir.	$(p_i)_3$ tangent dir.	$(p_i)_4$ parameter
0	$3/\lambda^2$	$-\mu^2$	$-(\mu/\lambda)^2$	$2(1.5)^{-8}$
1	$1/\lambda^2$	$-0.1$	$-0.5$	$2(1.5)^{-7}$
2	$1/\lambda^2$	$-0.1$	$-1$	$2(1.5)^{-6}$
3	$1/\lambda^2$	$-0.1$	$-1$	$2(1.5)^{-5}$
4	$1/\lambda^2$	$-0.1$	$-1$	$2(1.5)^{-4}$
5	$1/\lambda^2$	$-0.1$	$-1$	$2(1.5)^{-3}$
6	$1/\lambda^2$	$-0.1$	$-1$	$2(1.5)^{-2}$
7	$1/\lambda^2$	$-0.1$	$-1$	$2(1.5)^{-1}$
8	$0.5/\lambda^2$	$-1$	$-1$	$2$
9	$100/\lambda^2$	$-0.1$	$100(\mu/\lambda)^2$	$-2$
10	$40/\lambda^2$	$-0.1$	$(\mu/\lambda)^2$	$-2(1.5)^{-1}$
11	$10/\lambda^2$	$-0.1$	$(\mu/\lambda)^2$	$-2(1.5)^{-2}$
12	$1/\lambda^2$	$-0.1$	$(\mu/\lambda)^2$	$-2(1.5)^{-3}$
13	$1/\lambda^2$	$-0.1$	$(\mu/\lambda)^2$	$-2(1.5)^{-4}$
14	$1/\lambda^2$	$-0.1$	$(\mu/\lambda)^2$	$-2(1.5)^{-5}$
15	$0.3/\lambda^2$	$-0.1$	$(\mu/\lambda)^2$	$-2(1.5)^{-6}$

Table 2: Coefficients of the quadratic forms on the sets  $N_0, \dots, N_{15}$ , where  $\lambda = 3.858169402$ ,  $\mu = 0.07775708341$  are approximate eigenvalues of  $DH_{a_0}(z_0)$ .

computational point of view, it is better to make the matrix  $V$  somehow uniform, i.e. such that the coefficients on the diagonal are of the same magnitude. This can be achieved by setting coefficients  $\alpha_i \approx \lambda_i^{-2}$  for these  $i$  such that  $|\lambda_i| \gg 1$ .

This can be seen in the first column of the Table 2 and the second part of the third column.

The coefficients for  $N_9$  are chosen to be able to switch from the unstable to the stable manifolds.

Notice also a different choice of the coefficients for the quadratic form  $Q_0$  in  $N_0$ . The reason is that we have to prove that the center-unstable manifold at the fixed point  $c_0$  is a horizontal disk in  $N_0$  satisfying the cone conditions. Therefore we need to apply the method described in Theorem 7 to the inverse of  $PH$ . In this case, the coefficients in the second and the third column correspond to expanding directions with approximate eigenvalues  $1/\mu$  and  $\lambda/\mu$ , both greater than 1.

We have

**Lemma 16** *The cone conditions are satisfied for the sequence of covering relations*

$$N_0 \xrightarrow{PH} N_1 \xrightarrow{PH} \dots \xrightarrow{PH} N_{15}.$$

**Proof:** Observe that verification of the cone conditions for these covering relations require computation of the  $DPH$  which involves the second order derivatives of  $H$ . In the computer assisted proof we computed an enclosure for

$DPH(N_i)$ ,  $i = 0, 1, \dots, 14$  using whole  $N_i$  without subdivision as an initial condition of the routine which computes  $DPH$ . Then we applied Lemma 14 and Lemma 15 to prove that the cone conditions are satisfied. The C++ program which verifies the assertion executes within less than one second on a laptop-type computer. ■

### 6.3 Parameterization of center-unstable and center-stable manifolds at $c_0$ and $c_{15}$ , respectively.

In this section we will use the method described in Theorem 7 in order to parameterize the center-unstable and center-stable manifolds as a horizontal and vertical discs satisfying the cone conditions in  $N_0$  and  $N_{15}$ , respectively. This, together with Lemma 13 and Lemma 16 will give us a proof of Theorem 12.

Put

$$\begin{aligned}\widetilde{N}_0 &= (z_0, [u_0]) + \widetilde{M} \cdot ((d_0)_1, (d_0)_2, (d_0)_3), \\ \widetilde{N}_{15} &= (z_0, [s_0]) + \widetilde{M} \cdot ((d_{15})_1, (d_{15})_2, (d_{15})_3),\end{aligned}$$

where  $\widetilde{M}$  is a  $3 \times 3$  minor of the matrix  $M_{15} = M_0$  (see (17)) after removing the last column and the last row. The coefficients  $(d_i)_j$  are listed in the Table 1. Geometrically, these sets are just projection onto  $(x, y, t)$  coordinates of  $N_0$  and  $N_{15}$ , respectively.

The set  $\widetilde{N}_{15}$  is a three-dimensional h-set with two expanding directions (corresponding to eigenvalues  $\lambda$  and  $\frac{\lambda}{\mu}$ ) and one nominally stable direction (corresponding to the eigenvalue  $\mu$ ), where  $\lambda, \mu$  are eigenvalues of  $DH_{a_0}(z_0)$  – see (21).

On the set  $\widetilde{N}_0$  we will compute the inverse map of  $PH_a$  so that the role of nominally stable and nominally unstable directions interchange. Hence, the set  $\widetilde{N}_0$  has two nominally unstable directions (with eigenvalues  $\mu^{-1}$  and  $\lambda/\mu$ ) and one nominally stable direction (with eigenvalue  $\lambda^{-1}$ ).

On both sets we set quadratic forms defining the cones to be equal to

$$\widetilde{Q}_0 = -\Delta((p_0)_1, (p_0)_2, (p_0)_3), \quad (22)$$

$$\widetilde{Q}_{15} = \Delta((p_{15})_1, (p_{15})_2, (p_{15})_3), \quad (23)$$

where the coefficients are listed in the Table 2.

**Lemma 17** *The center-stable manifold of  $PH$  at  $c_{15}$  can be parameterized as a vertical disk in  $N_{15}$  satisfying the cone conditions with respect to the quadratic form  $Q_{15}$ .*

**Proof:** We use Theorem 7. Let  $\pi_a$  denote a projection onto the parameter coordinate. With a computer assistance we verified that for  $a \in \pi_a(N_{15})$  holds

$$\widetilde{N}_{15} \xrightarrow{PH_a} \widetilde{N}_{15}$$

and the cone conditions are satisfied with the quadratic form  $\widetilde{Q}_{15}$ . Hence, the first assumption of Theorem 7 is fulfilled.

Then we computed the constants  $A$ ,  $M$ ,  $L$  which appear in assumptions 2-3 of Theorem 7 and we obtained

$$\begin{aligned} A &\geq 0.099394300936541294, \\ M &\leq 0.084042214456891598, \\ L &\leq 0.0070394636406844067. \end{aligned}$$

Hence constant  $\Gamma$  from assumption 4 of Theorem 7 can be chosen to be equal to  $\Gamma = 0.57737423322563175$ . With this  $\Gamma$  the coefficient  $\delta$  defined in (9) is equal to

$$\delta = \frac{\Gamma^2}{\|\alpha\|} \leq 16.54078540195168.$$

where  $\alpha$  appears in the decomposition of the quadratic form  $\widetilde{Q}_{15}(x, y, t) = \alpha(x, t) - \beta(y)$  and  $\|\alpha\| = \max \{0.3/\lambda^2, (\mu/\lambda)^2\} = 0.3/\lambda^2$  – see Table 2.

From Theorem 7 it follows that the center-stable manifold of  $PH$  at  $c_{15}$  can be parameterized as a vertical disk in  $N_{15}$  satisfying the cone conditions with respect to the quadratic form

$$\overline{Q}_{15}(x, y, t, a) = \delta \widetilde{Q}_{15}(x, y, z) - a^2.$$

Recall that by  $Q_{15}$  we denote the quadratic form on  $N_{15}$ . To finish the proof let us observe that

$$Q_{15}(x, y, t, a) = \widetilde{Q}_{15}(x, y, t) - 2(1.5)^{-6}a^2$$

see (23) and Table 2. Moreover, we have  $2(1.5)^{-6}\delta > 1$ . This shows that

$$\delta Q_{15}(x, y, t, a) = \delta \left( \widetilde{Q}_{15}(x, y, t) - 2(1.5)^{-6}a^2 \right) < \overline{Q}_{15}(x, y, t, a).$$

Therefore the center-stable manifold of  $PH$  at  $c_{15}$  is a vertical disk in  $N_{15}$  satisfying the cone condition for the quadratic form  $Q_{15}$ . ■

We have a similar lemma about the parameterization of the center-unstable manifold of  $PH$  at  $c_0$  as a horizontal disk in  $N_0$ .

**Lemma 18** *The center-unstable manifold of  $PH$  at  $c_0$  can be parameterized as a horizontal disk in  $N_0$  satisfying the cone conditions with respect to the quadratic form  $Q_0$ .*

**Proof:** We will proceed as in Lemma 17 but for the map  $PH^{-1}$ . With a computer assistance we verified that  $a \in \pi_a(N_0)$  holds

$$\widetilde{N}_0 \xrightarrow{PH_a^{-1}} \widetilde{N}_0$$

and the cone conditions are satisfied. We computed the constants which appear in (8–9) and we got

$$\begin{aligned} A &\geq 0.1877584261322994, \\ M &\leq 0.2795983187542756, \\ L &\leq 0.015049353557694945, \\ \Gamma &= 0.33278415598142302, \\ \delta &\leq 18.316620936531205. \end{aligned}$$

Hence, the center-stable manifold for  $PH^{-1}$  at  $c_0$  is a vertical disk in  $N_0$  satisfying the cone condition for the quadratic form

$$\overline{Q_0(x, y, t, a)} = \delta \widetilde{Q_0}(x, y, t) - a^2.$$

From (22) and Table 2 we have

$$Q_0(x, y, t, a) = -\widetilde{Q_0}(x, y, t, a) + 2(1.5)^{-8}a^2.$$

Since  $2(1.5)^{-8}\delta > 1$  we have

$$-\delta Q_0(x, y, t, a) = \delta \left( \widetilde{Q_0}(x, y, t, a) - 2(1.5)^{-8}a^2 \right) < \overline{Q_0(x, y, t, a)}$$

and the center-stable manifold of  $PH^{-1}$  at  $c_0$  can be parameterized as a vertical disk in  $N_0$  satisfying the cone conditions with respect to the quadratic form  $-Q_0$ . This means that the center-unstable manifold of  $PH$  can be parameterized as a horizontal disk in  $N_0$  satisfying the cone conditions with respect to the quadratic form  $Q_0$ . ■

**Proof of Theorem 12:** We obtain our conclusion from Lemmas 16, 17 and 18 combined with the discussion of the strategy of the proof in the first part of Section 4. ■

## 7 Application to the forced damped pendulum equation.

Let us consider an equation for the forced damped pendulum motion

$$\ddot{x} + \beta \dot{x} + \sin(x) = \cos(t). \quad (24)$$

The equation (24) is a well known example of an equation which exhibits chaotic dynamics see [GH] and references given there. In [BCGH] it has been proven (a computer assisted proof) that for the parameter  $\beta = 0.1$  the  $2\pi$ -time map is semiconjugated to the full shift on three symbols on some compact invariant set.

The equation (24) defines a flow on  $\mathbb{R}^2 \times S^1$ , where  $S^1$  is a unit circle. Let us define the Poincaré map  $T_\beta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$T_\beta(x, \dot{x}) = (x(2\pi), \dot{x}(2\pi)), \quad (25)$$

where  $x(t)$  is a solution of (24) with the parameter  $\beta$ .

The aim of this section is to prove the following theorem.

**Theorem 19** *For all parameter values*

$$\beta \in \mathcal{B} = 0.247133729485 + [-1, 1] \cdot 1.2 \cdot 10^{-10}$$

*there exists a hyperbolic fixed point for  $T_\beta$  corresponding to a  $2\pi$  periodic solution of (24). Moreover, there exists a parameter value  $\beta \in \mathcal{B}$  such that the map  $T_\beta$  has a quadratic homoclinic tangency unfolding generically for that fixed point.*

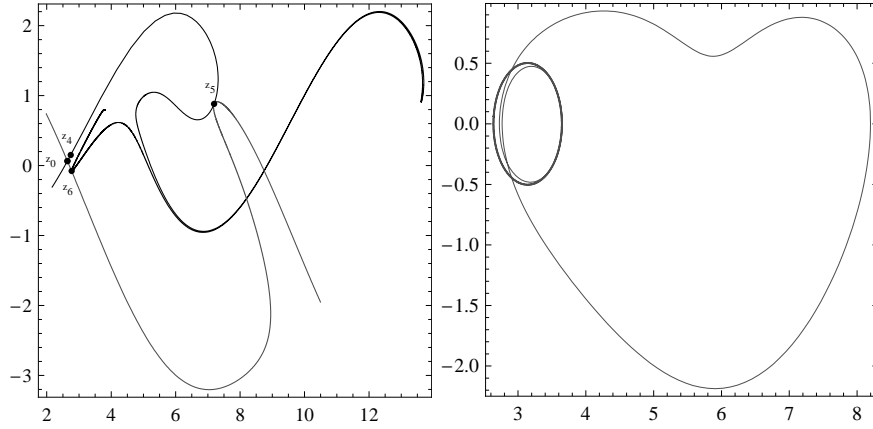


Figure 2: Left: parts of the unstable and stable manifolds of a fixed point of  $T_\beta$ , for  $\beta = 0.247133729485$ . Right: Periodic orbit and a homoclinic orbit to the periodic orbit for (24) projected onto  $(x, \dot{x})$  coordinates. This homoclinic orbit corresponds to the quadratic tangency of invariant manifolds. The left picture suggests also the existence of transversal homoclinic points.

The proof of Theorem 19 uses the same method as for the Hénon map case. Since all the details has been discussed in Section 6 we will give here only the definition of the sets, quadratic forms and we will state the numerical lemmas.

The main difference between the Hénon map case and that of the forced damped pendulum equation is how we compute derivatives of a map up to the second order, required in the proof. Since the Hénon map is given explicitly these derivatives can be computed by hand or using the Automatic Differentiation tools [G]. For the forced damped pendulum equation we used the  $C^r$ -Lohner algorithm presented in [WZ2]. This algorithm allows us to integrate efficiently the variational equations for ODE's and compute partial derivatives of Poincaré maps.

## 7.1 Heteroclinic chain of covering relations.

Let  $PT_\beta : P\mathbb{R}^2 \rightarrow P\mathbb{R}^2$  denote the map induced by  $T_\beta$  on the projective bundle and let  $\psi$  be a local parameterization of the space  $P\mathbb{R}^2$  as defined in (13). Let

$$PT : P\mathbb{R}^2 \times \mathbb{R} \ni (x, \dot{x}, t, \beta) = ((\psi^{-1} \circ PT_\beta \circ \psi)(x, \dot{x}, t), \beta) \in P\mathbb{R}^2 \times \mathbb{R}.$$

Let

$$\begin{aligned} \beta_0 &= 0.247133729485, \\ z_0 &= (2.6410109874338904, 0.063471204982120187), \\ u_0 &= (0.76818278871270265, 0.64023058590290362), \end{aligned} \quad (26)$$

$$s_0 = (0.67655372773981033, -0.73639327365298942), \quad (27)$$

$$\lambda = 211.83022271012155, \quad (28)$$

$$\mu = 0.00099918347695168025. \quad (29)$$

From numerical experiments we got that  $z_0$  (see Fig. 2) is an approximate fixed point for  $T_{\beta_0}$  with approximate eigenvalues  $\lambda$ ,  $\mu$  and approximate eigenvectors  $u_0$ ,  $s_0$ , respectively. This point has been found by the standard Newton method.

Let us denote

$$\begin{aligned} z_1 &= z_0 + 1.41442890240556 \cdot 10^{-8} u_0, & c_0 &= \psi^{-1}((z_0, u_0), \beta_0), \\ z_i &= T_{\beta_0}(z_{i-1}), \text{ for } i = 2, 3, 4, 5, & c_1 &= \psi^{-1}((z_1, u_0), \beta_0) \\ z_8 &= z_0 + 19.0992395815 \cdot 10^{-8} s_0, & c_i &= PT(c_{i-1}), \text{ for } i = 2, 3, 4, 5, \\ z_i &= T_{\beta_0}^{-1}(z_{i+1}) \text{ for } i = 7, 6, & c_8 &= \psi^{-1}((z_8, s_0), \beta_0) \\ z_9 &= z_0, & c_i &= PT^{-1}(c_{i-1}) \text{ for } i = 7, 6, \\ & & c_9 &= \psi^{-1}((z_0, s_0), \beta_0). \end{aligned}$$

Some of these points are shown in Fig. 2. The points  $z_1$  and  $z_8$  are chosen close to a heteroclinic trajectory for  $T_{\beta_0}$ . Numerical simulation shows that

$$PT^{-1}(c_6) - c_5 \approx (3.42 \cdot 10^{-12}, 6.34 \cdot 10^{-12}, -2.32 \cdot 10^{-8}, 0). \quad (30)$$

In fact, we observe that between  $c_5$  and  $c_6$  the role of the tangent direction changes and at  $c_6$  it becomes repelling.

The reason for which we define two points  $z_1$  and  $z_8$  and compute their forward and backward trajectory, respectively, is due to the numerical problems with forward propagation of tangent coordinate at  $z_6$ ,  $z_7$ , even in simulation only. Recall, the tangent coordinate  $t$  becomes strongly repelling at  $z_6$  and  $z_7$  with eigenvalue of the order  $10^5$ . Therefore, it is easier to propagate the preimage of the stable direction at  $z_8$  which is attracting at  $z_7$  and  $z_6$  for the inverse map.

Approximate estimation (30) shows us that the points  $c_i$ ,  $i = 1, \dots, 8$  are close to the possible existing heteroclinic trajectory for  $PT$ .

Now we will define the coordinate systems of the h-sets centered at  $c_i$ 's. Let  $M_i$  be the matrix of coordinate system of the set centered at  $c_i$ . We assume  $M_i$  has the form (17). These matrices are computed as follows

$i$	$10^{10} \cdot (d_i)_1$ unstable dir.	$10^{10} \cdot (d_i)_2$ stable dir.	$10^8 \cdot (d_i)_3$ tangent dir.	$10^{10} \cdot (d_i)_4$ parameter
0	5	0.2	0.26	$1.2(1.01)^5$
1	0.4	6	2	$1.2(1.01)^4$
2	0.4	6	4	$1.2(1.01)^3$
3	0.6	6	2	$1.2(1.01)^2$
4	5	2	1	$1.2(1.01)$
5	27	2	1	1.2
6	0.4	15	0.4	$1.2(1.01)$
7	0.4	8	0.4	$1.2(1.01)^2$
8	0.4	10	1.2	$1.2(1.01)^3$
9	0.4	5	0.2	$1.2(1.01)^4$

Table 3: Diameters of the h-sets in the heteroclinic sequence for the  $PT$  map. The h-sets  $N_0, \dots, N_5$  have two unstable directions given by  $u_i$  and the parameter. The h-sets  $N_6, \dots, N_9$  have two unstable directions given by  $u_i$  and the tangent coordinate.

- put  $u_9 = u_0$ ,  $u_1 = u_0$ ,  $u_8 = u_0$  and  $s_9 = s_0$ ,  $s_1 = s_0$ ,  $s_8 = s_0$ , where  $u_0, s_0$  are defined by (26–27),
- for  $i = 2, 3, 4, 5$  we set

$$\begin{aligned}
u_i &= \pi_t(c_i), \\
s_i &= \pi_t(PT_{\beta_0}^{-1}(z_{i+1}, \pi_t(c_{i+1})^\perp)), i = 2, 3, 4, \\
s_5 &= u_5^\perp,
\end{aligned}$$

- for  $i = 6, 7$  we set

$$\begin{aligned}
s_i &= \pi_t(c_i), \\
u_i &= \pi_t\left(PT_{\beta_0}\left(T^{-1}(z_i), \pi_t\left(PT_{\beta_0}^{-1}(z_i, s_i)\right)^\perp\right)\right).
\end{aligned}$$

As in the case of the Hénon map the h-sets  $N_0, \dots, N_9$  for the map  $PT$  will be defined by formula (20) with matrices  $M_i$  as above and the diameters given in Table 3.

The h-sets  $N_0, \dots, N_5$  have two unstable directions given by  $u_i$  and the parameter. The h-sets  $N_6, \dots, N_9$  have two unstable directions given by  $u_i$  and the tangent coordinate.

Now, we can state the following numerical lemma

**Lemma 20** *The map  $PT$  is well defined and continuous on  $\bigcup_{i=0}^9 N_i$ . Moreover, the following covering relations hold true*

$$N_0 \xrightarrow{PT} N_1 \xrightarrow{PT} \dots \xrightarrow{PT} N_9. \quad (31)$$

**Proof:** In the computer assisted proof of the above lemma we used the  $C^1$ -Lohner algorithm [Z] and the CAPD library [CAPD] in order to integrate the variational equations for (24) and to compute the map  $PT$ . We used the Taylor method of the order 20 with a variable time step. To verify the inequalities required for the covering relations we subdivided the boundary of each h-set with a grid depending on the set under consideration. The total number of boxes we used is 5546. The C++ program which verifies the existence of covering relations (31) executes within 18 seconds on a computer with the Intel Xeon 5160, 3GHz processor. ■

## 7.2 The cone conditions along the heteroclinic chain of covering relations for the map $PT$ .

Recall, that for a  $p \in \mathbb{R}^n$  by  $\Delta(p)$  we denoted a diagonal matrix with  $p_i$ 's on the diagonal.

For  $i = 0, \dots, 9$  we define the quadratic form on the h-set  $N_i$  by

$$Q_i = \Delta((p_i)_1, (p_i)_2, (p_i)_3, (p_i)_4)$$

where the coefficients are listed in Table 4.

Let us comment about the choice of these coefficients. In the example presented in Section 5 there is no dependency on the parameter, hence the cone conditions are easily achievable. In general case this dependency has huge influence for the choice of the parameters in quadratic forms corresponding to the parameter variable. In fact, this sometimes forces large scaling (not only by some small factor), like in the Hénon map case).

The other constraints on the coefficients are related to the parameterization of center-unstable and center-stable manifolds in the first and last sets in the heteroclinic chain of covering relations.

The main constraint, however, appears for the covering relation in the switch between manifolds. Here it is necessary to set relatively large coefficients corresponding to both unstable variables in the set after this switch and small coefficient for the stable variable in the main phase space. This can be seen in Tables 2 and 4 for the sets  $N_9$  and  $N_6$ , respectively. In the next sets we can use the hyperbolicity of the map to make these coefficients more uniform and to reach constraints at the begin and at the end of the chain.

For  $i = 2, \dots, 9$  the coordinate systems on  $N_i$  are given by the matrices  $M_i$  used to define the sets  $N_i$ , respectively.

We have the following numerical result

**Lemma 21** *The cone conditions are satisfied for the sequence of covering relations*

$$N_0 \xrightarrow{PT} N_1 \xrightarrow{PT} \dots \xrightarrow{PT} N_9.$$

**Proof:** By Lemma 14 it is enough to verify that for  $i = 0, \dots, 8$  the interval matrix

$$V_i = [DPT(N_i)]_I^T Q_{N_{i+1}} [DPT(N_i)]_I - Q_{N_i}$$

$i$	$(p_i)_1$ unstable dir.	$(p_i)_2$ stable dir.	$(p_i)_3$ tangent dir.	$(p_i)_4$ parameter
0	$80/\lambda^2$	$-\mu^2$	$-(\mu/\lambda)^2$	$(1.1)^{-5}$
1	$1/\lambda^2$	$-0.01$	$-10^{-7}$	$(1.1)^{-4}$
2	$1/\lambda^2$	$-1$	$-10^{-5}$	$(1.1)^{-3}$
3	$1/\lambda^2$	$-1$	$-10^{-5}$	$(1.1)^{-2}$
4	$1/\lambda^2$	$-1$	$-10^{-5}$	$(1.1)^{-1}$
5	$10/\lambda^2$	$-1$	$-10^{-5}$	3
6	$1000/\lambda^2$	$-10^{-4}$	$10^6(\mu/\lambda)^2$	$-(1.1)^{-1}$
7	$1/\lambda^2$	$-1$	$(\mu/\lambda)^2$	$-(1.1)^{-2}$
8	$1/\lambda^2$	$-1$	$(\mu/\lambda)^2$	$-(1.1)^{-3}$
9	$1/\lambda^2$	$-1$	$(\mu/\lambda)^2$	$-(1.1)^{-4}$

Table 4: Coefficients of the quadratic forms on the sets  $N_0, \dots, N_9$ , where  $\lambda, \mu$  are approximate eigenvalues of  $DT_{\beta_0}(z_0)$  and are given in (28-29).

is positive definite, where  $DPT$  is computed in the coordinate systems of  $N_i$  and  $N_{i+1}$ . Notice, to compute  $DPT$  we need second order derivatives of the map  $T$ . We used the  $C^2$ -Lohner algorithm [WZ2] and the CAPD library [CAPD] in order to integrate the second order variational equations for (24). We used the Taylor method of the order 20 and a variable time step. No subdivision of the sets  $N_i$  were necessary, i.e. whole sets  $N_i$  were used as an initial condition for the routine which computes  $DPT$ . The C++ program which verifies the cone conditions from this lemma executes within 1 seconds a computer with the Intel Xeon 5160, 3GHz processor. ■

### 7.3 Parameterization of center-unstable and center-stable manifolds at $c_0$ and $c_9$ , respectively.

**Lemma 22** *The following statements hold true.*

- *The center-stable manifold of  $PT$  at  $c_9$  can be parameterized as a vertical disk in  $N_9$  satisfying the cone conditions with respect to the quadratic form  $Q_9$ .*
- *The center-unstable manifold of  $PT$  at  $c_0$  can be parameterized as a vertical disk in  $N_0$  satisfying the cone conditions with respect to the quadratic form  $Q_0$ .*

The computer assisted proof of the above lemma is essentially the same as the proof of Lemma 17 and Lemma 18 for the Hénon map. Therefore we skip the details. The C++ program which verifies the assertion executes within 11 seconds on the Intel Xeon 5160, 3GHz processor. The most time-consuming part is the verification of the existence of covering relations for the projected sets in three-dimensional space.

We used the Taylor method of the order 20 and a variable time step when check the covering relations and when integrate second order variational equations.

**Proof of Theorem 19:** We conclude the proof as in the Hénon map example. ■

## 8 Implementation notes.

In order to compute bounds for the Hénon map and the Poincaré map  $T_\beta$  and their derivatives we used the interval arithmetic [M], automatic differentiation [G] and the  $C^r$ -Lohner algorithm [WZ2] developed at the Jagiellonian University by the CAPD group [CAPD]. The C++ source files of the program with an instruction how it should be compiled and run are available at [W].

The program has been tested under several linux distributions, including 32 and 64 bits architectures and the gcc compiler versions 4.1.2, 4.2.1 and 4.3.2 on the Intel Pentium IV, Intel Core 2 Duo, Intel Xeon and the AMD Quad Core processors.

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